

# SMOOTHING EFFECT OF WEAK SOLUTIONS FOR THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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ABSTRACT. In this paper, we consider the spatially homogeneous Boltzmann equation without angular cutoff. We prove that every  $L^1$  weak solution to the Cauchy problem with finite moments of all order acquires the  $C^\infty$  regularity in the velocity variable for the positive time.

## 1. INTRODUCTION

Consider the Cauchy problem for the spatially homogeneous Boltzmann equation,

$$(1.1) \quad \begin{cases} f_t(t, v) = Q(f, f)(t, v), & t \in \mathbb{R}^+, v \in \mathbb{R}^3, \\ f(0, v) = f_0(v), \end{cases}$$

where  $f = f(t, v)$  is the density distribution function of particles with velocity  $v \in \mathbb{R}^3$  at time  $t$ . The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*,$$

which is well-defined for suitable functions  $f$  and  $g$  specified later. Notice that the collision operator  $Q(\cdot, \cdot)$  acts only on the velocity variable  $v \in \mathbb{R}^3$ . In the following discussion, we will use the  $\sigma$ -representation, that is, for  $\sigma \in \mathbb{S}^2$ ,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which give the relations between the post and pre collisional velocities. For mono-atomic gas, the non-negative cross section  $B(z, \sigma)$  depends only on  $|z|$  and the scalar product  $\frac{z}{|z|} \cdot \sigma$ . As in [5, 6, 7], we assume that it takes the form

$$(1.2) \quad B(v - v_*, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

in which it contains a kinetic factor given by

$$(1.3) \quad \Phi(|v - v_*|) = \Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma,$$

with  $\gamma > -3$  and a factor related to the collision angle with singularity,

$$(1.4) \quad b(\cos \theta)\theta^{2+2s} \rightarrow K, \quad \text{when } \theta \rightarrow 0+,$$

for some positive constant  $K$  and  $0 < s < 1$ .

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The main purpose of this paper is to show the smoothing effect of the spatially homogeneous Boltzmann equation, that is, any weak solution to the Cauchy problem (1.1) acquires regularity as soon as  $t > 0$ . Let us recall the precise definition of weak solution for the Cauchy problem (1.1) given in [15], see also [16]. To this end, we introduce the standard notation,

$$\|f\|_{L_\ell^p} = \left( \int_{\mathbb{R}^3} |f(v)|^p (1 + |v|)^{\ell p} dv \right)^{1/p}, \quad \text{for } p \geq 1, \ell \in \mathbb{R},$$

$$\|f\|_{L \log L} = \int_{\mathbb{R}^3} |f(v)| \log(1 + |f(v)|) dv.$$

**Definition 1.1.** Let  $f_0 \geq 0$  be a function defined on  $\mathbb{R}^3$  with finite mass, energy and entropy, that is,

$$\int_{\mathbb{R}^3} f_0(v) [1 + |v|^2 + \log(1 + f_0(v))] dv < +\infty.$$

$f$  is a weak solution of the Cauchy problem (1.1), if it satisfies the following conditions:

$$\begin{aligned} f &\geq 0, \quad f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^3)) \cap L^1([0, T]; L_{2+\gamma}^1(\mathbb{R}^3)), \\ f(0, \cdot) &= f_0(\cdot), \\ \int_{\mathbb{R}^3} f(t, v) \psi(v) dv &= \int_{\mathbb{R}^3} f_0(v) \psi(v) dv \quad \text{for } \psi = 1, v_1, v_2, v_3, |v|^2; \\ f(t, \cdot) &\in L \log L, \quad \int_{\mathbb{R}^3} f(t, v) \log f(t, v) dv \leq \int_{\mathbb{R}^3} f_0 \log f_0 dv, \quad \forall t \geq 0; \\ \int_{\mathbb{R}^3} f(t, v) \varphi(t, v) dv &- \int_{\mathbb{R}^3} f_0(v) \varphi(0, v) dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau, v) \partial_\tau \varphi(\tau, v) dv \\ &= \int_0^t d\tau \int_{\mathbb{R}^3} Q(f, f)(\tau, v) \varphi(\tau, v) dv, \end{aligned}$$

where  $\varphi \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$ . Here, the right hand side of the last integral given above is defined by

$$\begin{aligned} &\int_{\mathbb{R}^3} Q(f, f)(v) \varphi(v) dv \\ &= \frac{1}{2} \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} B f(v_*) f(v) (\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*)) dv dv_* d\sigma. \end{aligned}$$

Hence, this integral is well defined for any test function  $\varphi \in L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^3))$  (see p. 291 of [15]).

To state the main theorem in this paper, we introduce the entropy dissipation functional by

$$D(g, f) = - \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B (g'_* f' - g_* f) \log f dv dv_* d\sigma,$$

where  $f = f(v)$ ,  $f' = f(v')$ ,  $g_* = g(v_*)$ ,  $g'_* = g(v'_*)$ .

**Theorem 1.2.** Let the cross section  $B$  in the form (1.2) satisfy (1.3) and (1.4) with  $0 < s < 1$ .

1) Suppose that  $\gamma > \max\{-2s, -1\}$ . Let  $f$  be a weak solution of the Cauchy problem (1.1). For  $0 \leq T_0 < T_1$ , if  $f$  satisfies

$$(1.5) \quad |v|^\ell f \in L^\infty([T_0, T_1]; L^1(\mathbb{R}^3)) \quad \text{for any } \ell \in \mathbb{N},$$

then

$$f \in L^\infty([t_0, T_1]; \mathcal{S}(\mathbb{R}^3)),$$

for any  $t_0 \in ]T_0, T_1[$ .

2) When  $-1 \geq \gamma > -2s$ , the same conclusion as above holds if we have the following entropy dissipation estimate

$$(1.6) \quad \int_{T_0}^{T_1} D(f(t), f(t)) dt < \infty.$$

The existence of weak solutions to the Cauchy problem (1.1) was proved by Villani [15] when  $\gamma \geq -2$ , assuming additionally in the case  $\gamma > 0$  that  $f_0 \in L^1_{2+\delta}$  for some  $\delta > 0$ . One important property of the weak solution for the hard potentials (namely when  $\gamma > 0$ ) is, according to the work by Wennberg [17] (cf. also Bobylev[8]), the moment gain property. It means that  $f$  satisfies (1.5) for arbitrary  $T_0 > 0$  when the initial data only satisfies finite mass, energy and entropy. However, without assuming the moment condition (1.5), we can still consider the smoothing effect in case with mild singularity ( $0 < s < 1/2$ ) for the hard potential ( $\gamma > 0$ ), and the argument is similar to the one used in [12] (see Theorem 5.2 in Section 5).

This kind of regularization property has been studied by many authors, cf. [2, 3, 10, 12, 13, 14]. However, to our knowledge, it has not yet been completely established in the sense that the kinetic factor  $\Phi(|z|)$  was modified to avoid the singularity at the origin except the Maxwellian molecule case in previous works, and moreover some extra conditions other than those in Definition 1.1 of weak solution were required in [3, 10].

We would like to emphasize that the result of Theorem 1.2 gives the full regularization property for any weak solution satisfying some natural boundedness condition in some weighted  $L^1$  and  $L \log L$  space, that requires no differentiation on the solution.

Recently in [11], it was proved that  $W_p^{1,1} \cap H^3$  (strong) solutions gain full regularity in the case  $0 < s < 1/2$ . Their method is based on the a priori estimate of the smooth solution, together with results given in [9] about the propagation of the norm  $W_p^{1,1}$  and the uniqueness of the solution. Different from [11], we start from the weak solution given in Definition 1.1 without any known uniqueness result. Therefore, a priori estimate for the smooth function is not enough to show the regularity for the weak solution in  $L^1$  with moments. For the proof of Theorem 1.2, some suitable mollifier, acting to the weak solution, becomes necessary, so that its commutator with the collision operator requires some subtle analysis.

Throughout this paper, we will use the following notations:  $f \lesssim g$  means that there exists a generic positive constant  $C$  such that  $f \leq Cg$ ; while  $f \gtrsim g$  means  $f \geq Cg$ . And  $f \sim g$  means that there exist two generic positive constant  $c_1$  and  $c_2$  such that  $c_1 f \leq g \leq c_2 g$ .

The rest of the paper will be organized as follows. In the next section, we will prove a uniform coercivity estimate that improves the one given in [1] which has its own interest. The mollifier and the commutator estimate will be given in Section

3. In Section 4 we prove the smoothing effect of weak solution with extra  $L^2$  assumption. The last section is devoted to the proof of Theorem 1.2.

## 2. A UNIFORM COERCIVE ESTIMATE

In this section, we will improve the coercive estimate for the collision operator obtained in [1] by removing the restriction on  $v$  in a bounded domain.

In view of the definition of the weak solution, for  $D_0, E_0 > 0$  we set

$$\mathcal{U}(D_0, E_0) = \{g \in L^1_2 \cap L \log L ; g \geq 0, \|g\|_{L^1} \geq D_0, \|g\|_{L^1_2} + \|g\|_{L \log L} \leq E_0\}.$$

Set  $B(R) = \{v \in \mathbb{R}^3 ; |v| \leq R\}$  for  $R > 0$  and set  $B_0(R, r) = \{v \in B(R) ; |v - v_0| \geq r\}$  for a  $v_0 \in \mathbb{R}^3$  and  $r \geq 0$ . It follows from the definition of  $\mathcal{U}(D_0, E_0)$  that there exist positive constants  $R > 1 > r_0$  depending only on  $D_0, E_0$  such that

$$(2.1) \quad g \in \mathcal{U}(D_0, E_0) \text{ implies } \chi_{B_0(R, r_0)} g \in \mathcal{U}(D_0/2, E_0),$$

where  $\chi_A$  denotes a characteristic function of the set  $A \subset \mathbb{R}^3$ . In fact, noting that for  $R, M > 0$

$$R^2 \int_{\{|v| > R\}} g dv + \log(1 + M) \int_{\{g > M\}} g dv \leq E_0.$$

We have

$$\int_{\{|v| \leq R\} \cap \{g \leq M\}} g dv \geq 3D_0/4$$

if  $R \geq 2\sqrt{2E_0/D_0}$  and  $\log(1 + M) \geq 8E_0/D_0$ , moreover we have

$$\int_{\{|v - v_0| < r_0\} \cap \{g \leq M\}} g dv \leq D_0/4$$

if  $r_0 \leq (3D_0/(16\pi \exp(8E_0/D_0)))^{1/3}$ .

**Proposition 2.1.** *Suppose that the cross section  $B$  of the form (1.2) satisfies (1.3) and (1.4) with  $0 < s < 1$  and  $\gamma > -3$ . If  $D_0, E_0 > 0$  and if  $g \in \mathcal{U}(D_0, E_0)$  then there exist positive constants  $c_0, C$  depending only on  $D_0, E_0$  such that for any  $f \in \mathcal{S}(\mathbb{R}^3)$ ,*

$$(2.2) \quad -\left(Q(g, f), f\right)_{L^2} \geq c_0 \|\langle v \rangle^{\gamma/2} f\|_{H^s}^2 - C \|\langle v \rangle^{\gamma/2} f\|_{H^{(-\gamma/2)+}}^2,$$

where  $a^+ = \max\{a, 0\}$  for  $a \in \mathbb{R}$ . Furthermore, if  $\gamma + 2s \leq 0$ ,  $0 < s' < s$  and if  $g$  belongs to  $L^{3/(3+\gamma+2s')}_{-\gamma}$  then there exists a  $C_1 > 0$  independent of  $g$  such that for any  $f \in \mathcal{S}(\mathbb{R}^3)$ ,

$$(2.3) \quad -\left(Q(g, f), f\right)_{L^2} \geq c_0 \|\langle v \rangle^{\gamma/2} f\|_{H^s}^2 - (C + C_1 \|g\|_{L^{3/(3+\gamma+2s')}_{-\gamma}}) \|\langle v \rangle^{\gamma/2} f\|_{H^{s'}}^2.$$

**Remark 2.2.** *It should be noted that the above coercive estimate is more precise than Theorem 1.2 of [11] and more adaptable to prove the regularity of weak solutions. In fact, the coercive estimate (2.2) is uniform with respect to  $g$ . If  $\gamma + 4s > 0$  and  $D(g, g) < \infty$  then  $g$  belongs to  $L^{3/(3+\gamma+2s')}_{-\gamma}$ , provided that  $g \in L^1_\ell$  for a sufficiently large  $\ell$ . In fact, it follows from the proof of Corollary 2.4 below that  $D(g, g) < \infty$  implies  $\sqrt{g} \in H^s_{\gamma/2}$  and hence  $\langle v \rangle^\gamma g \in L^{3/(3-2s)}$  by means of the Sobolev embedding theorem, which together with Lemma 3.8 below lead us to this conclusion.*

*Proof.* Put

$$\mathcal{C}_\gamma(g, f) = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} b(\cdot) |v - v_*|^\gamma g_*(f' - f)^2 dv dv_* d\sigma,$$

and note that

$$(Q(g, f), f) = -\frac{1}{2} \mathcal{C}_\gamma(g, f) + \frac{1}{2} \iiint \Phi b g_*(f'^2 - f^2) dv dv_* d\sigma.$$

It follows from the Cancellation Lemma and Remark 6 in [1] that

$$\begin{aligned} \left| \iiint b |v - v_*|^\gamma g_*(f^2 - f'^2) dv dv_* d\sigma \right| &\lesssim \left| \iiint |v - v_*|^\gamma g_* f^2 dv dv_* \right| \\ &\lesssim \|g\|_{L^1_{|\gamma|}} \|f\|_{H_{\gamma/2}^{(-\gamma/2)^+}}^2, \end{aligned}$$

where the last inequality in the case  $\gamma \geq 0$  is trivial. While  $\gamma < 0$ , this follows from the fact that

$$|v - v_*|^\gamma \lesssim \langle v \rangle^\gamma \{ \mathbf{1}_{|v - v_*| \geq \langle v \rangle/2} + \mathbf{1}_{|v - v_*| < \langle v \rangle/2} \langle v_* \rangle^{-\gamma} |v - v_*|^\gamma \},$$

and the Hardy inequality  $\sup_{v_*} \int |v - v_*|^\gamma |F(v)|^2 dv \lesssim \|F\|_{H^{-\gamma/2}}^2$  for  $F = \langle v \rangle^{\gamma/2} f$ . Furthermore, it follows from the Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned} \left| \iiint |v - v_*|^\gamma g_* f^2 dv dv_* \right| &\lesssim \|g\|_{L^1} \|F\|_{L^2}^2 + \iint \frac{\langle v_* \rangle^{|\gamma|} g(v_*) F(v)^2}{|v - v_*|^{-\gamma}} dv dv_* \\ &\lesssim \|g\|_{L^1} \|F\|_{L^2}^2 + \|\langle v \rangle^{|\gamma|} g\|_{L^{3/(3+\gamma+2s')}} \|F^2\|_{L^{3/(3-2s')}} \lesssim \|g\|_{L^{3/(3+\gamma+2s')}} \|f\|_{H_{\gamma/2}^{s'}}^2, \end{aligned}$$

where we have used the Sobolev embedding in the last inequality.

For the proof of the proposition, it now suffices to consider only the quantity  $\mathcal{C}_\gamma(g, f)$ . The case  $\gamma = 0$  is obvious. In fact, by Corollary 3 and Proposition 2 in [1], there exists a  $c_0 = c_0(D_0, E_0) > 0$  depending only on  $D_0, E_0 > 0$  such that

$$(2.4) \quad \mathcal{C}_0(g, f) \geq c_0 \int_{\{|\xi| \geq 1\}} \left| |\xi|^s \hat{f}(\xi) \right|^2 d\xi, \quad \forall f \in \mathcal{S}(\mathbb{R}^3),$$

where  $\hat{f}(\xi)$  is the Fourier transform of  $f$  with respect to the variable  $v \in \mathbb{R}^3$ . From the proof in [1], it should be noted that (2.4) holds for any  $f \in L^2$  such that the left hand side is finite.

We consider the case  $\gamma \neq 0$ , following the argument used in the proof of Lemma 2 of [1]. Choose  $R, r_0$  such that (2.1) holds. Let  $\varphi_R$  be a non-negative smooth function not greater than one, which is 1 for  $|v| \geq 4R$  and 0 for  $|v| \leq 2R$ . In view of

$$\frac{\langle v \rangle}{4} \leq |v - v_*| \leq 2\langle v \rangle \quad \text{on } \text{supp } (\chi_{B(R)})_* \varphi_R,$$

we have

$$\begin{aligned} 4^{|\gamma|} \Phi(|v - v_*|) g_*(f' - f)^2 &\geq (g \chi_{B(R)})_* (\langle v \rangle^{\gamma/2} \varphi_R)^2 (f' - f)^2 \\ &\geq (g \chi_{B(R)})_* \left[ \frac{1}{2} \left( (\langle v \rangle^{\gamma/2} \varphi_R f)' - \langle v \rangle^{\gamma/2} \varphi_R f \right)^2 - \left( (\langle v \rangle^{\gamma/2} \varphi_R)' - \langle v \rangle^{\gamma/2} \varphi_R \right)^2 f'^2 \right]. \end{aligned}$$

It follows from the mean value theorem that for a  $\tau \in (0, 1)$

$$\begin{aligned} \left| (\langle v \rangle^{\gamma/2} \varphi_R)' - \langle v \rangle^{\gamma/2} \varphi_R \right| &\lesssim \langle v + \tau(v' - v) \rangle^{\gamma/2-1} |v - v_*| \sin \frac{\theta}{2} \\ &\lesssim \langle v_* \rangle^{|\gamma/2-1|} \langle v' - v_* \rangle^{\gamma/2} \sin \frac{\theta}{2} \\ &\lesssim \langle v_* \rangle^{|\gamma/2|+|\gamma/2-1|} \langle v' \rangle^{\gamma/2} \sin \frac{\theta}{2}, \end{aligned}$$

because  $|v - v_*|/\sqrt{2} \leq |v' - v_*| \leq |v + \tau(v' - v) - v_*| \leq |v - v_*|$  for  $\theta \in [0, \pi/2]$ . Therefore, we have

$$(2.5) \quad \mathcal{C}_\gamma(g, f) \geq 2^{-1-2|\gamma|} \mathcal{C}_0(g \chi_{B(R)}, \varphi_R \langle v \rangle^{\gamma/2} f) - C_R \|g\|_{L^1} \|f\|_{L_{\gamma/2}^2}^2,$$

for a positive constant  $C_R \sim R^{|\gamma|+|\gamma-2|}$ . For a set  $B(4R)$  we take a finite covering

$$B(4R) \subset \bigcup_{v_j \in B(4R)} A_j, \quad A_j = \{v \in \mathbb{R}^3; |v - v_j| \leq \frac{r_0}{4}\}.$$

For each  $A_j$  we choose a non-negative smooth function  $\varphi_{A_j}$  which is 1 on  $A_j$  and 0 on  $\{|v - v_j| \geq r_0/2\}$ . Note that

$$\frac{r_0}{2} \leq |v - v_*| \leq 6R \text{ on } \text{supp} (\chi_{B_j(R, r_0)})_* \varphi_{A_j}.$$

Then we have

$$\begin{aligned} \Phi(|v - v_*|) g_* (f' - f)^2 &\gtrsim \min\{r_0^{\gamma^+}, R^{-(\gamma)^+}\} (g \chi_{B_j(R, r_0)})_* \varphi_{A_j}^2 (f' - f)^2 \\ &\gtrsim R^{-\gamma^+} \min\{r_0^{\gamma^+}, R^{-(\gamma)^+}\} (g \chi_{B_j(R, r_0)})_* \\ &\quad \times \left[ \frac{1}{2} \left( (\langle v \rangle^{\gamma/2} \varphi_{A_j} f)' - \langle v \rangle^{\gamma/2} \varphi_{A_j} f \right)^2 - \left( (\langle v \rangle^{\gamma/2} \varphi_{A_j})' - \langle v \rangle^{\gamma/2} \varphi_{A_j} \right)^2 f'^2 \right]. \end{aligned}$$

Since  $\left| (\langle v \rangle^{\gamma/2} \varphi_{A_j})' - \langle v \rangle^{\gamma/2} \varphi_{A_j} \right| \lesssim R^{|\gamma|+1} \langle v' \rangle^\gamma \sin \theta/2$  if  $|v_*| \leq R$ , we obtain

$$(2.6) \quad \mathcal{C}_\gamma(g, f) \gtrsim \min\{(r_0/R)^{\gamma^+}, R^{-(\gamma)^+}\} \mathcal{C}_0(g \chi_{B_j(R, r_0)}, \varphi_{A_j} \langle v \rangle^{\gamma/2} f) - C'_{R, r_0} \|g\|_{L^1} \|f\|_{L_{\gamma/2}^2}^2,$$

for a positive constant  $C'_{R, r_0} \sim R^{2+2|\gamma|}$ . It follows from (2.4), (2.5) and (2.6) that there exist  $c'_0, C, C' > 0$  depending only on  $D_0, E_0$  such that

$$(2.7) \quad \begin{aligned} \mathcal{C}_\gamma(g, f) &\geq c'_0 \left( \| \langle D \rangle^s \varphi_R \langle v \rangle^{\gamma/2} f \|^2 + \sum_j \| \langle D \rangle^s \varphi_{A_j} \langle v \rangle^{\gamma/2} f \|^2 \right) - C \|f\|_{L_{\gamma/2}^2}^2 \\ &\geq c'_0 \| \langle v \rangle^{\gamma/2} f \|_{H^s}^2 - C' \|f\|_{L_{\gamma/2}^2}^2, \end{aligned}$$

because  $\varphi_R^2 + \sum_j \varphi_{A_j}^2 \geq 1$  and commutators  $[\langle D \rangle^s, \varphi_R], [\langle D \rangle^s, \varphi_{A_j}]$  are  $L^2$  bounded operators.  $\square$

**Remark 2.3.** (2.7) holds for any  $f \in L_{\gamma/2}^2$  such that  $\mathcal{C}_\gamma(g, f)$  is finite, because of the remark after (2.4). Similarly, (2.2) holds for any  $f \in L_{\gamma/2}^2$  if  $\gamma \geq 0$  and if its left hand side is finite.

**Corollary 2.4.** *Let  $f(t) \in L^1_{\max\{2,\gamma\}} \cap L \log L$  be a weak solution. Suppose that the cross section  $B$  is the same as in Proposition 2.1. Assume that for a  $T > 0$  we have*

$$(2.8) \quad \int_0^T D(f(\tau), f(\tau)) d\tau < \infty.$$

*Then there exist positive constants  $c_f$  and  $C_f > 0$  such that*

$$(2.9) \quad c_f \int_0^T \|\sqrt{f(\tau)}\|_{H^s_{\gamma/2}}^2 d\tau \leq \int_0^T D(f(\tau), f(\tau)) d\tau + C_f \int_0^T \|f(\tau)\|_{L^1_{\gamma+}} d\tau.$$

*Proof.* We first consider the case  $\gamma < 0$ . Note

$$\begin{aligned} D(f, f) &= - \iiint B(f'_* f' - f_* f) \log f dv dv_* d\sigma \\ &= \frac{1}{4} \iiint B(f'_* f' - f_* f) (\log f'_* f' - \log f_* f) dv dv_* d\sigma \\ &\geq \frac{1}{4} \iiint b(\cdot) \langle v - v_* \rangle^\gamma (f'_* f' - f_* f) (\log f'_* f' - \log f_* f) dv dv_* d\sigma, \end{aligned}$$

because  $(x - y)(\log x - \log y) \geq 0$  and  $\Phi(|v - v_*|) \geq \langle v - v_* \rangle^\gamma$ . Then we have

$$\begin{aligned} D(f, f) &\geq - \iiint b(\cdot) \langle v - v_* \rangle^\gamma (f'_* f' - f_* f) \log f dv dv_* d\sigma \\ &= \iiint b(\cdot) \langle v - v_* \rangle^\gamma f_* \left( f \log \frac{f}{f'} - f + f' \right) dv dv_* d\sigma \\ &\quad + \iiint b(\cdot) \langle v - v_* \rangle^\gamma f_* (f - f') dv dv_* d\sigma \\ &\geq \iiint b(\cdot) \langle v - v_* \rangle^\gamma f_* \left( \sqrt{f'} - \sqrt{f} \right)^2 dv dv_* d\sigma - C \|f\|_{L^1}^2, \end{aligned}$$

where we have used  $x \log(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2$  and the Cancellation Lemma in the last inequality, as the same as in the proof of Theorem 1 in [1]. Since the proof of Proposition 2.1 still works with  $\Phi$  replaced by  $\langle v - v_* \rangle^\gamma$ , we obtain the desired estimate in view of Remark 2.3. The case  $\gamma \geq 0$  is easier because we do not need to replace  $\Phi$  by  $\langle v - v_* \rangle^\gamma$  when Cancellation Lemma is applied.  $\square$

### 3. MOLLIFIER AND COMMUTATOR ESTIMATE

Since the weak solution is only in  $L^1$ , we can not use it directly as a test function in the definition of weak solution to get the energy estimate. To overcome this, we need to mollify it by some suitable mollifiers so that to consider the commutators between the mollifiers and the collision operator becomes necessary.

Let  $\lambda, N_0 \in \mathbb{R}$ ,  $\delta > 0$  and put

$$(3.1) \quad M_\lambda^\delta(\xi) = \frac{\langle \xi \rangle^\lambda}{(1 + \delta \langle \xi \rangle)^{N_0}}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

Then  $M_\lambda^\delta(\xi)$  belongs to the symbol class  $S_{1,0}^{\lambda-N_0}$  of pseudo-differential operators and belongs to  $S_{1,0}^\lambda$  uniformly with respect to  $\delta \in ]0, 1]$ .  $M_\lambda^\delta(D_v)$  denotes the associated pseudo-differential operator. By direct calculation we see that for any  $\alpha$  there exists a  $C_\alpha > 0$  independent of  $\delta$  such that

$$(3.2) \quad \left| \partial_\xi^\alpha M_\lambda^\delta(\xi) \right| \leq C_\alpha M_\lambda^\delta(\xi) \langle \xi \rangle^{-|\alpha|}.$$

**Lemma 3.1.** *There exists a constant  $C > 0$  independent of  $\delta$  such that*

$$(3.3) \quad \begin{aligned} & |M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*)| \\ & \leq C \langle \xi \rangle^\lambda \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} + CM_\lambda^\delta(\xi - \xi_*) \left\{ \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} + \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \mathbf{1}_{|\xi|/2 > \langle \xi_* \rangle} \right\} \\ & + CM_\lambda^\delta(\xi - \xi_*) \left( \frac{M_\lambda^\delta(\xi_*) (1 + \delta \langle \xi - \xi_* \rangle)^{N_0}}{\langle \xi - \xi_* \rangle^\lambda} \right) \mathbf{1}_{\sqrt{2}|\xi| > \langle \xi_* \rangle \geq |\xi|/2}. \end{aligned}$$

And if  $p \geq N_0 - \lambda$

$$(3.4) \quad \begin{aligned} & |M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*)| \leq CM_\lambda^\delta(\xi - \xi_*) \left\{ \left( \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^p \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \right. \\ & \left. + \left( \frac{M_\lambda^\delta(\xi_*) (1 + \delta \langle \xi - \xi_* \rangle)^{N_0}}{\langle \xi - \xi_* \rangle^\lambda} + 1 \right) \mathbf{1}_{\sqrt{2}|\xi| > \langle \xi_* \rangle \geq |\xi|/2} + \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \mathbf{1}_{|\xi|/2 > \langle \xi_* \rangle} \right\}. \end{aligned}$$

*Proof.* We first note

$$(3.5) \quad \begin{cases} \langle \xi \rangle \lesssim \langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle, & \text{on supp } \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|}, \\ \langle \xi \rangle \sim \langle \xi - \xi_* \rangle, & \text{on supp } \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2}, \\ \langle \xi \rangle \sim \langle \xi_* \rangle \gtrsim \langle \xi - \xi_* \rangle, & \text{on supp } \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2}. \end{cases}$$

Since  $\langle \xi \rangle^p M_\lambda^\delta(\xi)$  is increasing function of  $\langle \xi \rangle$ , we have

$$\langle \xi \rangle^p M_\lambda^\delta(\xi) \lesssim \langle \xi_* \rangle^p M_\lambda^\delta(\xi_*) \sim \langle \xi_* \rangle^p M_\lambda^\delta(\xi - \xi_*) \text{ on supp } \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|},$$

and trivially,

$$M_\lambda^\delta(\xi) \leq \langle \xi \rangle^\lambda.$$

Note that

$$M_\lambda^\delta(\xi) \sim M_\lambda^\delta(\xi_*) \sim M_\lambda^\delta(\xi - \xi_*) \frac{M_\lambda^\delta(\xi_*) (1 + \delta \langle \xi - \xi_* \rangle)^{N_0}}{\langle \xi - \xi_* \rangle^\lambda} \text{ on supp } \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2}.$$

By the mean value theorem, we have

$$\begin{aligned} |M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*)| & \leq \int_0^1 |(\nabla_\xi M_\lambda^\delta)(\xi + \tau(\xi - \xi_*))| d\tau |\xi_*| \\ & \lesssim M_\lambda^\delta(\xi - \xi_*) \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \text{ on supp } \mathbf{1}_{\langle \xi_* \rangle \leq |\xi|/2}. \end{aligned}$$

Here we have used (3.2) and the second formula of (3.5). The above estimates imply (3.4) and (3.3).  $\square$

For the kinetic factor  $|v - v_*|^\gamma$ , we need to take into account the singular behavior close to  $|v - v_*| = 0$  except  $\gamma = 0$ . Therefore, we decompose the kinetic factor in two parts. Let  $0 \leq \phi(z) \leq 1$  be a smooth radial function with value 1 for  $z$  close to 0, and 0 for large values of  $z$ . Set

$$\Phi_\gamma(z) = \Phi_\gamma(z)\phi(z) + \Phi_\gamma(z)(1 - \phi(z)) = \Phi_c(z) + \Phi_{\bar{c}}(z).$$

And then correspondingly we can write

$$Q(f, g) = Q_c(f, g) + Q_{\bar{c}}(f, g),$$

where the kinetic factor in the collision operator is defined according to the decomposition respectively. Since  $\Phi_{\bar{c}}(z)$  is smooth, and  $\Phi_{\bar{c}}(z) \lesssim \tilde{\Phi}_\gamma(z)$ , where  $\tilde{\Phi}_\gamma(|z|) = (1 + |z|^2)^{\gamma/2}$  is the regular kinetic factor studied in [4]. Then  $Q_{\bar{c}}(f, g)$  has similar



properties as for  $Q_{\tilde{\Phi}_\gamma}(f, g)$  as regard to the upper bound and commutator estimations. We recall the Proposition 2.9 of [4].

**Proposition 3.2.** *Let  $\lambda \in \mathbb{R}$  and  $M(\xi)$  be a positive symbol of pseudo-differential operator in  $S_{1,0}^\lambda$  in the form of  $M(\xi) = \tilde{M}(|\xi|^2)$ . Assume that, there exist constants  $c, C > 0$  such that for any  $s, \tau > 0$*

$$c^{-1} \leq \frac{s}{\tau} \leq c \quad \text{implies} \quad C^{-1} \leq \frac{\tilde{M}(s)}{\tilde{M}(\tau)} \leq C,$$

and  $M(\xi)$  satisfies

$$|M^{(\alpha)}(\xi)| = |\partial_\xi^\alpha M(\xi)| \leq C_\alpha M(\xi) \langle \xi \rangle^{-|\alpha|},$$

for any  $\alpha \in \mathbb{N}^3$ . Then, if  $0 < s < 1/2$ , for any  $N > 0$  there exists a  $C_N > 0$  such that

$$(3.6) \quad |(M(D_v)Q_{\bar{c}}(f, g) - Q_{\bar{c}}(f, M(D_v)g), h)_{L^2}| \\ \leq C_N \|f\|_{L_{\gamma+}^1} \left( \|M(D_v)g\|_{L_{\gamma+}^2} + \|g\|_{H_{\gamma+}^{\lambda-N}} \right) \|h\|_{L^2}.$$

Furthermore, if  $1/2 < s < 1$ , for any  $N > 0$  and any  $\varepsilon > 0$ , there exists a  $C_{N,\varepsilon} > 0$  such that

$$(3.7) \quad |(M(D_v)Q_{\bar{c}}(f, g) - Q_{\bar{c}}(f, M(D_v)g), h)_{L^2}| \\ \leq C_{N,\varepsilon} \|f\|_{L_{(2s+\gamma-1)+}^1} \left( \|M(D_v)g\|_{H_{(2s+\gamma-1)+}^{2s-1+\varepsilon}} + \|g\|_{H_{\gamma+}^{\lambda-N}} \right) \|h\|_{L^2}.$$

When  $s = 1/2$  we have the same estimate as (3.7) with  $(2s + \gamma - 1)$  replaced by  $(\gamma + \kappa)$  for any small  $\kappa > 0$ .

**Remark 3.3.** In the case  $\gamma > 0$  and  $0 < s < 1/2$ , it follows from Lemma 3.1 of [12] and its proof that (3.6) can be replaced by

$$|(M(D_v)Q_{\bar{c}}(f, g) - Q_{\bar{c}}(f, M(D_v)g), h)_{L^2}| \\ \leq C_N \|f\|_{L_\gamma^1} \left( \|M(D_v)g\|_{L_{\gamma/2}^2} + \|g\|_{H_{\gamma/2}^{\lambda-N}} \right) \|h\|_{L_{\gamma/2}^2}.$$

From now on, we concentrate on the study for the singular part  $Q_c(f, g)$ .

**Proposition 3.4.** *Assume that  $0 < s < 1, \gamma + 2s > 0$ . Let  $0 < s' < s$  satisfy  $\gamma + 2s' > 0$  and  $2s' \geq (2s - 1)^+$ . If*

$$(3.8) \quad 5 + \gamma \geq 2(N_0 - \lambda),$$

then we have

1) If  $s' + \lambda < 3/2$ , then

$$\left| \left( M_\lambda^\delta(D_v) Q_c(f, g) - Q_c(f, M_\lambda^\delta(D_v)g), h \right) \right| \lesssim \|f\|_{L^1} \|M_\lambda^\delta(D_v)g\|_{H^{s'}} \|h\|_{H^{s'}}.$$

2) If  $s' + \lambda \geq 3/2$ , then

$$\left| \left( M_\lambda^\delta(D_v) Q_c(f, g) - Q_c(f, M_\lambda^\delta(D_v)g), h \right) \right| \\ \lesssim \left( \|f\|_{L^1} + \|f\|_{H^{(\lambda+s'-3)^+}} \right) \|M_\lambda^\delta(D_v)g\|_{H^{s'}} \|h\|_{H^{s'}}.$$

Furthermore, if  $s > 1/2$  and  $\gamma > -1$ , then the assumption (3.8) can be relaxed to

$$(3.9) \quad 4 + \gamma + 2s > 2(N_0 - \lambda).$$

*Proof.* For the proof we shall follow some of arguments from [5]. By using the formula from the Appendix of [1], we have

$$(Q_c(f, g), h) = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \\ \times \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma,$$

where  $\xi^- = \frac{1}{2}(\xi - |\xi|\sigma)$ . Therefore

$$\begin{aligned} & \left( M_\lambda^\delta(D) Q_c(f, g) - Q_c(f, M_\lambda^\delta(D) g), h \right) \\ &= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) [\hat{\Phi}_c(\xi_* - \xi^-) - \hat{\Phi}_c(\xi_*)] \\ & \quad \times \left( M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma \\ &= \iiint_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma + \iiint_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \cdots d\xi d\xi_* d\sigma \\ &= \mathcal{A}_1(f, g, h) + \mathcal{A}_2(f, g, h). \end{aligned}$$

Then, we write  $\mathcal{A}_2(f, g, h)$  as

$$\begin{aligned} \mathcal{A}_2 &= \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \hat{\Phi}_c(\xi_* - \xi^-) \cdots d\xi d\xi_* d\sigma \\ & \quad - \iiint b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \hat{\Phi}_c(\xi_*) \cdots d\xi d\xi_* d\sigma \\ &= \mathcal{A}_{2,1}(f, g, h) - \mathcal{A}_{2,2}(f, g, h). \end{aligned}$$

On the other hand, for  $\mathcal{A}_1$  we use the Taylor expansion of  $\hat{\Phi}_c$  of order 2 to have

$$\mathcal{A}_1 = \mathcal{A}_{1,1}(f, g, h) + \mathcal{A}_{1,2}(f, g, h),$$

where

$$\begin{aligned} \mathcal{A}_{1,1} &= \iiint b \xi^- \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} \left( M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \right) \\ & \quad \times \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_* d\sigma, \end{aligned}$$

and  $\mathcal{A}_{1,2}(f, g, h)$  is the remaining term corresponding to the second order term in the Taylor expansion of  $\hat{\Phi}_c$ .

We first consider  $\mathcal{A}_{1,1}$ . By writing

$$\xi^- = \frac{|\xi|}{2} \left( \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \frac{\xi}{|\xi|} - \sigma \right) + \left( 1 - \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2},$$

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on  $\mathbb{S}^2$ . Hence, we have

$$\mathcal{A}_{1,1} = \iint_{\mathbb{R}^6} K(\xi, \xi_*) \left( M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \overline{\hat{h}(\xi)} d\xi d\xi_*,$$

where

$$K(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left( 1 - \left( \frac{\xi}{|\xi|} \cdot \sigma \right) \right) \frac{\xi}{2} \cdot (\nabla \hat{\Phi}_c)(\xi_*) \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} d\sigma.$$

Note that  $|\nabla \hat{\Phi}_c(\xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+1}}$ , from the Appendix of [5]. If  $\sqrt{2}|\xi| \leq \langle \xi_* \rangle$ , then  $\sin(\theta/2) |\xi| = |\xi^-| \leq \langle \xi_* \rangle/2$  because  $0 \leq \theta \leq \pi/2$ , and we have

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi/2} \theta^{1-2s} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right).$$

On the other hand, if  $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$ , then

$$|K(\xi, \xi_*)| \lesssim \int_0^{\pi \langle \xi_* \rangle / (2|\xi|)} \theta^{1-2s} d\theta \frac{\langle \xi \rangle}{\langle \xi_* \rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s-1}.$$

Hence we obtain

$$(3.10) \quad |K(\xi, \xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right) \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s-1} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\}.$$

Similar to  $\mathcal{A}_{1,1}$ , we can also write

$$\mathcal{A}_{1,2} = \iint_{\mathbb{R}^6} \tilde{K}(\xi, \xi_*) \left( M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_*,$$

where

$$\tilde{K}(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \int_0^1 (1-\tau) (\nabla^2 \hat{\Phi}_c)(\xi_* - \tau \xi^-) \cdot \xi^- \cdot \xi^- \mathbf{1}_{|\xi^-| \leq \frac{1}{2}\langle \xi_* \rangle} d\tau d\sigma.$$

Again from the Appendix of [5], we have

$$|(\nabla^2 \hat{\Phi}_c)(\xi_* - \tau \xi^-)| \lesssim \frac{1}{\langle \xi_* - \tau \xi^- \rangle^{3+\gamma+2}} \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma+2}},$$

because  $|\xi^-| \leq \langle \xi_* \rangle/2$ , which leads to

$$(3.11) \quad |\tilde{K}(\xi, \xi_*)| \lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^2 \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} + \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\}.$$

It follows from (3.4) of Lemma 3.1, (3.10) and (3.11) that if  $p = N_0 - \lambda$ , then

$$|\mathcal{A}_1| \lesssim |\mathcal{A}_{1,1}| + |\mathcal{A}_{1,2}| \lesssim A_1 + A_2 + A_3,$$

where

$$(3.12) \quad A_1 = \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle \xi_* \rangle^{3+\gamma}} \right| \left| M_\lambda^\delta(\xi - \xi_*) \hat{g}(\xi - \xi_*) \right| |\hat{h}(\xi)| \left( \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{p-1} \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} d\xi_* d\xi,$$

and

$$\begin{aligned}
A_2 &= \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle \xi_* \rangle^{3+\gamma}} \right| \left| M_\lambda^\delta(\xi - \xi_*) \hat{g}(\xi - \xi_*) \right| |\hat{h}(\xi)| \\
&\quad \times \left( \frac{M_\lambda^\delta(\xi_*) (1 + (\delta \langle \xi - \xi_* \rangle)^{N_0})}{\langle \xi - \xi_* \rangle^\lambda} + 1 \right) \mathbf{1}_{\sqrt{2}|\xi| > \langle \xi_* \rangle \geq |\xi|/2} d\xi_* d\xi; \\
A_3 &= \iint_{\mathbb{R}^6} \left| \frac{\hat{f}(\xi_*)}{\langle \xi_* \rangle^{3+\gamma}} \right| \left| M_\lambda^\delta(\xi - \xi_*) \hat{g}(\xi - \xi_*) \right| |\hat{h}(\xi)| \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s-1} \mathbf{1}_{|\xi|/2 > \langle \xi_* \rangle} d\xi_* d\xi.
\end{aligned}$$

Putting  $\hat{G}(\xi) = \langle \xi \rangle^{s'} M_\lambda^\delta(\xi) \hat{g}(\xi)$  and  $\hat{H}(\xi) = \langle \xi \rangle^{s'} \hat{h}(\xi)$ , then we have

$$\begin{aligned}
|A_1|^2 &\lesssim \|\hat{f}\|_{L^\infty}^2 \left( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}_\xi^3} |\hat{H}(\xi)|^2 d\xi \right) \\
&\quad \times \left( \int_{\mathbb{R}^3} \frac{d\xi}{\langle \xi \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{3+\gamma-2(p-1)} \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} |\hat{G}(\xi - \xi_*)|^2 d\xi_* \right) \\
&\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2,
\end{aligned}$$

because  $\gamma + 2s' > 0$ , and  $3 + \gamma - 2(p-1) \geq 0$  from (3.8). Here we have used the fact that  $\langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle$  if  $\langle \xi_* \rangle \geq \sqrt{2}|\xi|$ .

We consider the case  $s > 1/2, \gamma > -1$ . For  $s > s' > 1/2$  we have

$$\begin{aligned}
|A_1|^2 &\lesssim \|\hat{f}\|_{L^\infty}^2 \left( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+1}} \int_{\mathbb{R}_\xi^3} |\hat{H}(\xi)|^2 d\xi \right) \\
&\quad \times \left( \int_{\mathbb{R}^3} \frac{d\xi}{\langle \xi \rangle^{3+\gamma+1}} \int_{\mathbb{R}^3} \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{3+\gamma+(2s'-1)-2(p-1)} \frac{\mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|}}{\langle \xi \rangle^{2(2s'-1)}} |\hat{G}(\xi - \xi_*)|^2 d\xi_* \right) \\
&\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2,
\end{aligned}$$

if  $3 + \gamma + (2s' - 1) - 2(p-1) > 0$ . Thus (3.8) can be relaxed to (3.9) to get the desired estimate for  $A_1$ . Here we remark that (3.8) or (3.9) are only required to estimate the part  $A_1$ .

Noting the third formula of (3.5), we get

$$\begin{aligned}
|A_2|^2 &\lesssim \left\{ \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2 d\xi_*}{\langle \xi_* \rangle^{6+2\gamma+2s'}} \int_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* \rangle} \left( \frac{\langle \xi_* \rangle^{2\lambda}}{\langle \xi - \xi_* \rangle^{2(\lambda+s')}} \right. \right. \\
&\quad \left. \left. + \frac{\langle \xi_* \rangle^{2(\lambda-N_0)}}{\langle \xi - \xi_* \rangle^{2(\lambda-N_0+s')}} + \frac{1}{\langle \xi - \xi_* \rangle^{2s'}} \right) d\xi \right\} \\
&\quad \times \left( \iint_{\mathbb{R}^6} |\hat{G}(\xi - \xi_*)|^2 |\hat{H}(\xi)|^2 d\xi d\xi_* \right).
\end{aligned}$$

If  $\lambda + s' < 3/2$ , then

$$\begin{aligned}
|A_2|^2 &\lesssim \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2}{\langle \xi_* \rangle^{3+2(\gamma+2s')}} d\xi_* \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \\
&\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2.
\end{aligned}$$

If  $\lambda + s' \geq 3/2$ , then

$$\begin{aligned} |A_2|^2 &\lesssim \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi_*)|^2 \langle \xi_* \rangle^{2(\lambda+s'+\varepsilon)}}{\langle \xi_* \rangle^{6+2(\gamma+2s')}} d\xi_* \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \\ &\lesssim \|f\|_{H^{\lambda+s'-3}}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2. \end{aligned}$$

Since  $2s' \geq 2s - 1$  and  $\gamma + 2s' > 0$ , we have

$$\begin{aligned} |A_3|^2 &\lesssim \|\hat{f}\|_{L^\infty}^2 \left( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} |\hat{H}(\xi)|^2 d\xi \right) \\ &\times \left( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} \left( \frac{\langle \xi_* \rangle}{\langle \xi \rangle} \right)^{2\{2s'-(2s-1)\}} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} |\hat{G}(\xi - \xi_*)|^2 d\xi \right) \\ &\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2. \end{aligned}$$

The above four estimates yield the desired estimate for  $\mathcal{A}_1(f, g, h)$ .

Next we consider  $\mathcal{A}_2(f, g, h) = \mathcal{A}_{2,1}(f, g, h) - \mathcal{A}_{2,2}(f, g, h)$ . The fact that  $|\xi^-| = |\xi| \sin(\theta/2) \geq \langle \xi_* \rangle/2$  and  $\theta \in [0, \pi/2]$  imply  $\sqrt{2}|\xi| \geq \langle \xi_* \rangle$ . Write

$$\mathcal{A}_{2,j} = \iint_{\mathbb{R}^6} K_j(\xi, \xi_*) \left( M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*) \right) \hat{f}(\xi_*) \hat{g}(\xi - \xi_*) \bar{\hat{h}}(\xi) d\xi d\xi_*.$$

Then we have

$$\begin{aligned} |K_2(\xi, \xi_*)| &= \left| \int b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \hat{\Phi}_c(\xi_*) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} d\sigma \right| \\ &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \frac{\langle \xi \rangle^{2s}}{\langle \xi_* \rangle^{2s}} \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle} \\ &\lesssim \frac{1}{\langle \xi_* \rangle^{3+\gamma}} \left\{ \mathbf{1}_{\sqrt{2}|\xi| \geq \langle \xi_* \rangle \geq |\xi|/2} + \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{2s} \mathbf{1}_{|\xi|/2 \geq \langle \xi_* \rangle} \right\}, \end{aligned}$$

which shows the desired estimate for  $\mathcal{A}_{2,2}$ , by exactly the same way as the estimation on  $A_2$  and  $A_3$ .

As for  $\mathcal{A}_{2,1}$ , it suffices to work under the condition  $|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2$ . In fact, on the complement of this set, we have  $|\xi_* - \xi^-| > |\xi_*|$ , and  $\hat{\Phi}_c(\xi_* - \xi^-)$  is the same as  $\hat{\Phi}_c(\xi_*)$ . Therefore, we consider  $\mathcal{A}_{2,1,p}$  which is defined by replacing  $K_1(\xi, \xi_*)$  by

$$K_{1,p}(\xi, \xi_*) = \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \hat{\Phi}_c(\xi_* - \xi^-) \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{|\xi_* \cdot \xi^-| \geq \frac{1}{2}|\xi^-|^2} d\sigma.$$

By noting

$$\mathbf{1} = \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi_* - \xi_*^- \rangle \leq 2\langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2} \mathbf{1}_{\langle \xi_* - \xi_*^- \rangle > 2\langle \xi_* - \xi^- \rangle} + \mathbf{1}_{\langle \xi_* \rangle < |\xi|/2},$$

we decompose respectively

$$\mathcal{A}_{2,1,p} = B_1 + B_2 + B_3.$$

On the sets for above integrals, we have  $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi_* \rangle$ , because  $|\xi^-| \lesssim |\xi_*|$  that follows from  $|\xi^-|^2 \leq 2|\xi_* \cdot \xi^-| \lesssim |\xi^-||\xi_*|$ . Furthermore, on the sets for  $B_1$  and  $B_2$  we have  $\langle \xi \rangle \sim \langle \xi_* \rangle$ , so that  $\langle \xi_* - \xi^- \rangle \lesssim \langle \xi \rangle$  and  $b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \mathbf{1}_{\langle \xi_* \rangle \geq |\xi|/2}$  is bounded.

Putting again  $\hat{G}(\xi) = \langle \xi \rangle^{s'} M_\lambda^\delta(\xi) \hat{g}(\xi)$  and  $\hat{H}(\xi) = \langle \xi \rangle^{s'} \hat{h}(\xi)$ , by Lemma 3.1 we have

$$\begin{aligned} |B_1|^2 &\lesssim \left[ \iiint \left| \frac{\hat{\Phi}_c(\xi_* - \xi^-)}{\langle \xi_* - \xi^- \rangle^{s'}} \right|^2 |\hat{f}(\xi_*)|^2 \right. \\ &\quad \times \left\{ M_\lambda^\delta(\xi_*)^2 \left( \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{2(s'+\lambda)}} + \frac{\delta^{2N_0} \mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{2(s'+\lambda-N_0)}} \right) \right. \\ &\quad \left. \left. + \frac{\mathbf{1}_{\langle \xi - \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle}}{\langle \xi - \xi_* \rangle^{2s'}} \right\} d\xi d\xi_* d\sigma \right] \left( \iiint |\hat{G}(\xi - \xi_*)|^2 |\hat{H}(\xi)|^2 d\sigma d\xi d\xi_* \right). \end{aligned}$$

Noting that  $\langle \xi_* \rangle \sim \langle \xi \rangle \sim \langle \xi^+ \rangle \lesssim \langle \xi^+ - u \rangle + \langle u \rangle$  with  $u = \xi_* - \xi^-$ , and moreover  $\langle u \rangle \lesssim \langle \xi_* \rangle$ , we see that if  $\lambda \geq 0$  then

$$M_\lambda^\delta(\xi_*)^2 \lesssim \frac{\langle \xi^+ - u \rangle^{2\lambda} + \langle u \rangle^{2\lambda}}{(1 + \delta \langle u \rangle)^{2N_0}}.$$

This is true even if  $\lambda < 0$ . Therefore, if  $s' + \lambda < 3/2$  we have

$$\begin{aligned} |B_1|^2 &\lesssim \|f\|_{L^1}^2 \int \langle u \rangle^{-(6+2\gamma+2s')} \\ &\quad \times \left\{ \int_{\langle \xi^+ - u \rangle \leq \langle u \rangle} \frac{(\langle \xi^+ - u \rangle^{2s'} + \langle u \rangle^{2\lambda})}{(1 + \delta \langle u \rangle)^{2N_0}} \left( \frac{1}{\langle \xi^+ - u \rangle^{2(s'+\lambda)}} + \frac{\delta^{2N_0}}{\langle \xi^+ - u \rangle^{2(s'+\lambda-N_0)}} \right) d\xi^+ \right. \\ &\quad \left. + \int_{\langle \xi^+ - u \rangle \leq \langle u \rangle} \frac{d\xi^+}{\langle \xi^+ - u \rangle^{2s'}} \right\} du \|M_\lambda^\delta(D)g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \\ &\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta(D)g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \int \frac{du}{\langle u \rangle^{3+2(\gamma+2s')}}. \end{aligned}$$

Here we have used the change of variables  $(\xi, \xi_*) \rightarrow (\xi^+, u)$  whose Jacobian is

$$\begin{aligned} \left| \frac{\partial(\xi^+, u)}{\partial(\xi, \xi_*)} \right| &= \left| \frac{\partial \xi^+}{\partial \xi} \right| = \frac{\left| I + \frac{\xi}{|\xi|} \otimes \sigma \right|}{8} \\ &= \frac{|1 + \frac{\xi}{|\xi|} \cdot \sigma|}{8} = \frac{\cos^2(\theta/2)}{4} \geq \frac{1}{8}, \quad \theta \in [0, \frac{\pi}{2}]. \end{aligned}$$

If  $s' + \lambda \geq 3/2$ , in view of  $\gamma + 2s' > 0$  we have

$$\begin{aligned} |B_1|^2 &\lesssim \int |\hat{f}(\xi_*)|^2 \left\{ \langle u \rangle^{2\lambda - (6+2\gamma+2s')} \log \langle u \rangle \right\} d\xi_* \|M_\lambda^\delta(D)g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \\ &\lesssim \|f\|_{H^{(\lambda+s'-3)+}}^2 \|M_\lambda^\delta(D)g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2, \end{aligned}$$

because  $\langle u \rangle \lesssim \langle \xi_* \rangle$  on the set of the integral.

As for  $B_2$ , we first note that, on the set of the integral,  $\xi^+ = \xi - \xi_* + u$  implies

$$\frac{\langle \xi - \xi_* \rangle}{2} \leq \langle \xi - \xi_* \rangle - |u| \leq \langle \xi^+ \rangle \leq \langle \xi - \xi_* \rangle + |u| \lesssim \langle \xi - \xi_* \rangle,$$

so that

$$(M_\lambda^\delta(\xi) \sim) M_\lambda^\delta(\xi^+) \sim M_\lambda^\delta(\xi - \xi_*).$$

and hence we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |B_2|^2 &\lesssim \|f\|_{L^1}^2 \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|}{\langle \xi_* - \xi^- \rangle^{2s'}} |\hat{G}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \iiint \frac{|\hat{\Phi}_c(\xi_* - \xi^-)|}{\langle \xi_* - \xi^- \rangle^{2s'}} |\hat{H}(\xi)|^2 d\sigma d\xi d\xi_* \\ &\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta(D)g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2, \end{aligned}$$

because  $\gamma + 2s' > 0$ .

On the set of integral for  $B_3$  we recall  $\langle \xi \rangle \sim \langle \xi - \xi_* \rangle$  and

$$|M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*)| \lesssim \frac{\langle \xi_* \rangle}{\langle \xi \rangle} M_\lambda^\delta(\xi - \xi_*),$$

so that

$$\begin{aligned} |B_3|^2 &\lesssim \|f\|_{L^1}^2 \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| \langle \xi_* \rangle}{\langle \xi \rangle^{2s'+1}} |\hat{G}(\xi - \xi_*)|^2 d\sigma d\xi d\xi_* \\ &\quad \times \iiint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| \langle \xi_* \rangle}{\langle \xi \rangle^{2s'+1}} |\hat{H}(\xi)|^2 d\sigma d\xi d\xi_*. \end{aligned}$$

We use the change of variables  $\xi_* \rightarrow u = \xi_* - \xi^-$ . Note that  $|\xi^-| \geq \frac{1}{2}\langle u + \xi^- \rangle$  implies  $|\xi^-| \geq \langle u \rangle / \sqrt{10}$ , and that

$$\langle \xi_* \rangle \lesssim \langle \xi_* - \xi^- \rangle + |\xi| \sin \theta / 2.$$

Then we have

$$\begin{aligned} \iint b \mathbf{1}_{|\xi^-| \geq \frac{1}{2}\langle \xi_* \rangle} \frac{|\hat{\Phi}_c(\xi_* - \xi^-)| \langle \xi_* \rangle}{\langle \xi \rangle^{2s'+1}} d\sigma d\xi_* &\lesssim \int \frac{\mathbf{1}_{\langle u \rangle \lesssim |\xi|}}{\langle u \rangle^{3+\gamma+2s'}} \left( \frac{\langle u \rangle}{\langle \xi \rangle} \right)^{2s'} \\ &\quad \times \left( \int b \mathbf{1}_{|\xi^-| \gtrsim \langle u \rangle} \frac{\langle u \rangle}{\langle \xi \rangle} d\sigma + \int b \sin(\theta/2) \mathbf{1}_{|\xi^-| \gtrsim \langle u \rangle} d\sigma \right) du, \end{aligned}$$

from which we also can obtain the desired bound for  $B_3$  if  $\gamma + 2s' > 0$ . In fact, the first integral on the sphere is bounded above by  $\langle u \rangle^{1-2s} / \langle \xi \rangle^{1-2s}$  and the second integral has the same bound when  $s > 1/2$ . On the other hand, the second integral is bounded by a constant when  $s < 1/2$  and by  $|\log(\langle \xi \rangle / \langle u \rangle)|$  when  $s = 1/2$ . The proof of 1) and 2) of the proposition is then completed.  $\square$

The combination of Proposition 3.4 and Proposition 3.2 together with its remark yield the following theorem.

**Theorem 3.5.** *Assume that  $0 < s < 1, \gamma + 2s > 0$ . Let  $0 < s' < s$  satisfy  $\gamma + 2s' > 0, 2s' \geq (2s - 1)^+$ . If a pair  $(N_0, \lambda)$  satisfies (3.8) then we have*

1) *If  $s' + \lambda < 3/2$ , we have*

$$\begin{aligned} &\left| \left( M_\lambda^\delta(D_v) Q(f, g) - Q(f, M_\lambda^\delta(D_v) g), h \right) \right| \\ &\lesssim \|f\|_{L^1_{\gamma^+ + (2s-1)^+}} \|M_\lambda^\delta(D_v) g\|_{H^{s'}_{\gamma^+ + (2s-1)^+}} \|h\|_{H^{s'}}. \end{aligned}$$

2) *If  $s' + \lambda \geq 3/2$ , we have*

$$\begin{aligned} &\left| \left( M_\lambda^\delta(D_v) Q(f, g) - Q(f, M_\lambda^\delta(D_v) g), h \right) \right| \\ &\lesssim \left( \|f\|_{L^1_{\gamma^+ + (2s-1)^+}} + \|f\|_{H^{(\lambda+s'-3)^+}} \right) \|M_\lambda^\delta(D_v) g\|_{H^{s'}_{\gamma^+ + (2s-1)^+}} \|h\|_{H^{s'}}. \end{aligned}$$

Furthermore, if  $s > 1/2$  and  $\gamma > -1$  then the same conclusion as above holds even when the condition (3.8) is replaced by (3.9). When  $0 < s < 1/2$  and  $\gamma > 0$ , we can use  $\|M_\lambda^\delta(D_v)g\|_{H_{\gamma/2}^{s'}} \|h\|_{H_{\gamma/2}^{s'}}$  for the corresponding terms in above estimates with smaller weight in the variable  $v$ .

We recall also the following upper bound estimate, Proposition 2.1 of [7], where we need the assumption  $\gamma + 2s > 0$  (see also Theorem 2.1 from [4]).

**Proposition 3.6.** *Let  $\gamma + 2s > 0$  and  $0 < s < 1$ . For any  $r \in [2s - 1, 2s]$  and  $\ell \in [0, \gamma + 2s]$  we have*

$$\left| \left( Q(f, g), h \right)_{L^2(\mathbb{R}^3)} \right| \lesssim \|f\|_{L_{\gamma+2s}^1} \|g\|_{H_{\gamma+2s-\ell}^r} \|h\|_{H_\ell^{2s-r}}.$$

In the following analysis, we shall need an interpolation inequality concerning weighted type Sobolev spaces in  $v$ , see for instance [10, 12].

**Lemma 3.7.** *For any  $k \in \mathbb{R}, p \in \mathbb{R}_+, \delta > 0$ ,*

$$\|f\|_{H_p^k(\mathbb{R}_v^3)}^2 \leq C_\delta \|f\|_{H_{2p}^{k-\delta}(\mathbb{R}_v^3)} \|f\|_{H_0^{k+\delta}(\mathbb{R}_v^3)}.$$

And also another interpolation in  $L^q$  is given by

**Lemma 3.8.** *Let  $1 < q < p$ . Assume that  $f \in L^p$  and  $\langle v \rangle^\ell f \in L^1$  for any  $\ell$ . Then  $\langle v \rangle^\ell f \in L^q$  for any  $\ell$ . More precisely, we have*

$$\|f\|_{L_\ell^q} \leq 2 \|f\|_{L^p}^{\frac{p(q-1)}{q(p-1)}} \|f\|_{L^1}^{\frac{p-q}{q(p-1)}}.$$

*Proof.* Take  $\lambda > 0$ , we rewrite

$$\begin{aligned} \|f\|_{L_\ell^q}^q &= \int_{\langle v \rangle^{\ell q} |f(v)|^{q-p} \leq \lambda} \langle v \rangle^{\ell q} |f(v)|^q dv + \int_{\langle v \rangle^{\ell q} |f(v)|^{q-p} > \lambda} \langle v \rangle^{\ell q} |f(v)|^q dv \\ &\leq \lambda \|f\|_{L^p}^p + \lambda^{\frac{q-1}{q-p}} \|f\|_{L^1}^{\frac{\ell q(p-1)}{p-q}}. \end{aligned}$$

Taking

$$\lambda = \|f\|_{L^1}^{\frac{(p-q)}{(p-1)}} \|f\|_{L^p}^{-\frac{p(p-q)}{(p-1)}},$$

we obtain the desired estimate.  $\square$

#### 4. SMOOTHING EFFECT OF $L^2$ WEAK SOLUTIONS

We start from a weak solution in  $L^2$  with bounded moments.

**Theorem 4.1.** *Assume that  $0 < s < 1$ ,  $\gamma + 2s > 0$ . If  $f$  belongs to  $L^\infty([t_0, T]; L_\ell^2(\mathbb{R}^3))$  for any  $\ell \in \mathbb{N}$  and is a non-negative weak solution of (1.1), then for any  $t_0 < \tilde{t}_0 < T$ , we have*

$$f \in L^\infty([\tilde{t}_0, T]; \mathcal{S}(\mathbb{R}^3)).$$

*Proof.* Without loss of generality, take  $t_0 = 0$ . Assume that, for  $a \geq 0$ , we have

$$(4.1) \quad \sup_{[0, T]} \|f(t, \cdot)\|_{H_\ell^a} < \infty \quad \text{for any } \ell \in \mathbb{N}.$$

Take  $\lambda(t) = Nt + a$  for  $N > 0$ . Choose  $N_0 = a + (5 + \gamma)/2$ . Then the pair  $(N_0, \lambda(t))$  satisfies (3.8). If we choose  $N, T_1 > 0$  such that  $NT_1 = (1 - s)$ , then

$$\lambda(T_1) - N_0 - a \leq \lambda(T_1) - N_0 < -3/2,$$



from which we have, for  $t, t' \in [0, T_1]$ ,

$$(4.2) \quad M_{\lambda(t)}^\delta f(t') \in L^\infty([0, T_1] \times [0, T_1]; H_\ell^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)),$$

because of (4.1). By the same way as in (3.5) and (3.6) of [13], we have

$$(4.3) \quad M_{\lambda(t)}^\delta f(t) \in C([0, T_1]; L^2(\mathbb{R}^3)),$$

and for any  $t \in ]0, T_1]$ , we have

$$(4.4) \quad \begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} (M_{\lambda(t)}^\delta f(t))^2 dv - \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} f(\tau) (\partial_\tau (M_{\lambda(\tau)}^\delta)^2) f(\tau) dv d\tau \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (M_{\lambda(0)}^\delta f_0)^2 dv \\ &+ \int_0^t \left( Q(f(\tau), M_{\lambda(\tau)}^\delta f(\tau)), M_{\lambda(\tau)}^\delta f(\tau) \right)_{L^2} d\tau \\ &+ \int_0^t \left( M_{\lambda(\tau)}^\delta Q(f(\tau), f(\tau)) - Q(f(\tau), M_{\lambda(\tau)}^\delta f(\tau)), M_{\lambda(\tau)}^\delta f(\tau) \right)_{L^2} d\tau, \end{aligned}$$

by taking  $(M_{\lambda(t)}^\delta)^2 f(t)$  as a test function in the definition of the weak solution, though it does not belong to  $L^\infty([0, T_1]; W^{2,\infty}(\mathbb{R}^3))$ . In fact, we can show (4.3) and (4.4) under a weaker condition than (4.2), which will be given in Lemma 4.3 below.

Noting

$$\partial_t M_{\lambda(t)}^\delta = N(\log \langle \xi \rangle) M_{\lambda(t)}^\delta,$$

by Theorem 3.5 we have

$$(4.5) \quad \begin{aligned} & \frac{1}{2} \| (M_{\lambda}^\delta f)(t) \|_{L^2}^2 \leq \frac{1}{2} \| f(0) \|_{H^a}^2 + \int_0^t \left( Q(f(\tau), (M_{\lambda}^\delta f)(\tau)), (M_{\lambda}^\delta f)(\tau) \right) d\tau \\ &+ C_f \int_0^t \| (M_{\lambda}^\delta f)(\tau) \|_{H_{(2s+\gamma-1)+}^{s'}} \| (M_{\lambda}^\delta f)(\tau) \|_{H^{s'}} d\tau \\ &+ CN \int_0^t \| (\log \langle D \rangle)^{1/2} (M_{\lambda}^\delta f)(\tau) \|_{L^2}^2 d\tau. \end{aligned}$$

Since the uniform coercive estimate (2.2) together with the interpolation in the Sobolev space yields

$$\left( Q(f(\tau), (M_{\lambda}^\delta f)(\tau)), (M_{\lambda}^\delta f)(\tau) \right) \leq -c_f \| (M_{\lambda}^\delta f)(\tau) \|_{H_{\gamma/2}^s}^2 + C_f \| f(\tau) \|_{H_{\gamma/2}^{-2}}^2,$$

by means of Lemma 3.7 we have

$$(4.6) \quad \| (M_{\lambda}^\delta f)(t) \|_{L^2}^2 + c_f \int_0^t \| (M_{\lambda}^\delta f)(\tau) \|_{H_{\gamma/2}^s}^2 d\tau \leq \| f(0) \|_{H^a}^2 + C_f \int_0^t \| f(\tau) \|_{H_\ell^a}^2 d\tau.$$

Taking  $\delta \rightarrow +0$  and  $t = T_1$ , we have  $f(T_1) \in H^{\lambda(T_1)} = H^{NT_1+a}$ . This is true for any  $0 < T_1 \leq T$ . Choosing  $N = (1-s)T_1^{-1}$ , we have that for any  $0 < T_1 \leq T$ ,

$$f(T_1) \in H^{(1-s)+a}.$$

Fix  $0 < s_0 < (1-s)$ . Then, by using Lemma 3.7 and assumption (4.1), we see that for any  $0 < t_1 < \tilde{t}_0$  and any  $\ell$ ,

$$\sup_{[t_1, T]} \| f(t, \cdot) \|_{H_\ell^{s_0+a}} < \infty.$$

We can restart by replacing  $a$  by  $a + s_0 = a_1$  and  $t_0$  by  $t_1$ . By induction, for  $a_0 = 0, a_k = k s_0$ , and  $t_k = \tilde{t}_0 - (2k)^{-1}(\tilde{t}_0 - t_0)$ , we have for any  $k \in \mathbb{N}$  and any  $\ell$ ,

$$f \in L^\infty([t_k, T]; H_\ell^{a_k}(\mathbb{R}^3)),$$

which concludes the proof of Theorem 4.1.  $\square$

**Remark 4.2.** When  $0 < s < 1/2$  and  $\gamma > 0$  we can use  $\int_0^t \|(M_\lambda^\delta f)(\tau)\|_{H_{\gamma/2}^{s'}}^2 d\tau$  for the corresponding term in (4.5). Hence, instead of (4.6), we can obtain

$$\|(M_\lambda^\delta f)(t)\|_{L^2}^2 \leq \|f(0)\|_{H^a}^2 + C_f \int_0^t \|f(\tau)\|_{H_{\gamma/2}^{-2}}^2 d\tau,$$

which shows that  $f(t) \in L^\infty([0, T]; L^2 \cap L_2^1(\mathbb{R}^3))$  implies  $f(t) \in H^\infty(\mathbb{R}^3)$  for  $t > 0$ .

**Lemma 4.3.** Let  $T_1 > 0$  and let  $M_{\lambda(t)}^\delta(\xi)$  be defined by (3.1) with  $\lambda = \lambda(t) = Nt + a$  for  $NT_1 < 1$  and  $a \in \mathbb{R}$ . Suppose that

$$f \in L^1([0, T_1]; L_{\max\{\gamma+2s, 2\}}^1(\mathbb{R}^3)) \cap L^\infty([0, T_1]; H^a(\mathbb{R}^3)).$$

If there exists  $s_1 > s$  such that

$$M_{\lambda(t)}^\delta f(t', v) \in L^\infty([0, T_1]_t \times [0, T_1]_{t'}; H_{\ell_0}^{s_1}(\mathbb{R}_v^3))$$

for  $\ell_0 = \max\{\gamma/2 + s, \gamma^+ + (2s - 1)^+\}$ , then we have (4.3), and (4.4) for any  $t \in [0, T_1]$ . Furthermore, if  $0 < s < 1/2$  and  $\gamma > 0$  we can take  $\ell_0 = \gamma/2 + s$ .

*Proof.* In Definition 1.1, taking  $\varphi(t, v) = \psi(v) \in C_0^\infty(\mathbb{R}^3)$ , we get

$$\int_{\mathbb{R}^3} f(t)\psi dv - \int_{\mathbb{R}^3} f(t')\psi dv = \int_{t'}^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau))\psi dv, \quad 0 \leq t' \leq t \leq T_0.$$

By taking a sequence  $\{\psi_j(v)\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}_v^3)$  such that  $(M_{\lambda(t)}^\delta)^{-1}\psi_j \rightarrow M_{\lambda(t)}^\delta f(t)$  in  $H_{\ell_0}^s$ , we can set  $\psi = (M_{\lambda(t)}^\delta)^2 f(t)$  for a fixed  $t$  because

$$\left| \int_{\mathbb{R}^3} f(t')(M_{\lambda(t)}^\delta)^2 f(t) dv \right| \leq \|M_{\lambda(t)}^\delta f(t')\|_{L^2} \|M_{\lambda(t)}^\delta f(t)\|_{L^2} < \infty,$$

and by noting

$$(Q(f, f), (M_\lambda^\delta)^2 f) = (Q(f, M_\lambda^\delta f), M_\lambda^\delta f) + (M_\lambda^\delta Q(f, f) - Q(f, M_\lambda^\delta f), M_\lambda^\delta f),$$

we have

$$\begin{aligned} & \left| \int_{t'}^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) (M_{\lambda(t)}^\delta)^2 f(t) dv \right| \\ & \lesssim \int_{t'}^t \|f(\tau)\|_{L_{\gamma+2s}^1} d\tau \left( \sup_{\tau, t \in [0, T_1]} \|M_{\lambda(t)}^\delta f(\tau)\|_{H_{\gamma/2+s}^s} \|M_{\lambda(t)}^\delta f(t)\|_{H_{\gamma/2+s}^s} \right) \\ & \quad + \left( \int_{t'}^t \|f(\tau)\|_{L_{\gamma^++(2s-1)^+}^1} d\tau + |t - t'| \sup_{\tau \in [0, T_1]} \|f(\tau)\|_{H^a} \right) \\ & \quad \times \left( \sup_{\tau, t \in [0, T_1]} \|M_{\lambda(t)}^\delta f(\tau)\|_{H_{\gamma^++(2s-1)^+}^s} \|M_{\lambda(t)}^\delta f(t)\|_{H^s} \right), \end{aligned}$$

thanks to Proposition 3.6 and Theorem 3.5. Setting  $\psi = (M_{\lambda(t')}^\delta)^2 f(t')$  also and taking the sum, we obtain

$$\begin{aligned}
 (4.7) \quad & \int_{\mathbb{R}^3} (M_{\lambda(t)}^\delta f(t))^2 dv - \int_{\mathbb{R}^3} (M_{\lambda(t')}^\delta f(t'))^2 dv \\
 &= \int_{\mathbb{R}^3} f(t) \left( (M_{\lambda(t)}^\delta)^2 - (M_{\lambda(t')}^\delta)^2 \right) f(t') dv \\
 & \quad + \int_{t'}^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) \left( (M_{\lambda(t)}^\delta)^2 f(t) + (M_{\lambda(t')}^\delta)^2 f(t') \right) dv.
 \end{aligned}$$

Since it follows from the mean value theorem that the first term on the right hand side is estimated by

$$|t - t'| \sup_{0 \leq t' < \bar{\tau} < t \leq T_1} \|M_{\lambda(\bar{\tau})}^\delta f(t)\|_{L^2} \|(\log \langle D \rangle) M_{\lambda(\bar{\tau})}^\delta f(t')\|_{L^2},$$

we obtain (4.3), namely  $M_{\lambda(t)}^\delta f(t) \in C([0, T_0]; L^2(\mathbb{R}^3))$ .

Taking  $\psi = (\log \langle D \rangle)^2 (M_{\lambda(t')}^\delta)^2 f(t')$ , we also have

$$(\log \langle D \rangle) M_{\lambda(t)}^\delta f(t) \in C([0, T_0]; L^2(\mathbb{R}^3)).$$

Taking the difference, instead of (4.7), we get

$$\begin{aligned}
 & \int_{\mathbb{R}^3} (M_{\lambda(t)}^\delta f(t))^2 dv + \int_{\mathbb{R}^3} (M_{\lambda(t')}^\delta f(t'))^2 dv \\
 &= \int_{\mathbb{R}^3} f(t) \left( (M_{\lambda(t)}^\delta)^2 + (M_{\lambda(t')}^\delta)^2 \right) f(t') dv \\
 & \quad + \int_{t'}^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) \left( (M_{\lambda(t)}^\delta)^2 f(t) - (M_{\lambda(t')}^\delta)^2 f(t') \right) dv,
 \end{aligned}$$

which shows

$$\lim_{t' \rightarrow t} \int_{\mathbb{R}^3} f(t) \left( (M_{\lambda(t)}^\delta)^2 + (M_{\lambda(t')}^\delta)^2 \right) f(t') dv = 2 \int_{\mathbb{R}^3} (M_{\lambda(t)}^\delta f(t))^2 dv,$$

and moreover

$$(4.8) \quad \lim_{t' \rightarrow t} \int_{\mathbb{R}^3} (M_{\lambda(t)}^\delta f(t)) (M_{\lambda(t')}^\delta f(t')) dv = \int_{\mathbb{R}^3} (M_{\lambda(t)}^\delta f(t))^2 dv.$$

To prove (4.4) we introduce

$$M_{\lambda(t)}^{\delta, \kappa} = \frac{M_{\lambda(t)}^\delta(\xi)}{1 + \kappa \langle \xi \rangle},$$

with a new parameter  $\kappa > 0$ . Divide  $[0, t]$  into  $k$  subintervals with the same length and put  $t_j = jt/k$  for  $j = 0, \dots, k$ . Similar to (4.7), we have

$$\begin{aligned}
 (4.9) \quad & \int_{\mathbb{R}^3} (M_{\lambda(t)}^{\delta, \kappa} f(t_j))^2 dv - \int_{\mathbb{R}^3} (M_{\lambda(t_{j-1})}^{\delta, \kappa} f(t_{j-1}))^2 dv \\
 &= \int_{\mathbb{R}^3} f(t_j) \left( (M_{\lambda(t_j)}^{\delta, \kappa})^2 - (M_{\lambda(t_{j-1})}^{\delta, \kappa})^2 \right) f(t_{j-1}) dv \\
 & \quad + \int_{t_{j-1}}^{t_j} d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau)) \left( (M_{\lambda(t_j)}^{\delta, \kappa})^2 f(t_j) + (M_{\lambda(t_{j-1})}^{\delta, \kappa})^2 f(t_{j-1}) \right) dv.
 \end{aligned}$$

Since we have

$$\begin{aligned}
& \int f(t_j) \left( (M_{\lambda(t_j)}^{\delta, \kappa})^2 - (M_{\lambda(t_{j-1})}^{\delta, \kappa})^2 \right) f(t_{j-1}) dv \\
&= \int 2N f(t_j) (\log \langle D \rangle) (M_{\lambda(t_j)}^{\delta, \kappa})^2 f(t_{j-1}) dv \quad \tau_j \in ]t_{j-1}, t_j[ \\
&= 2N \int (\sqrt{\log \langle D \rangle} M_{\lambda(t_j)}^{\delta, \kappa} f(t_j)) (\sqrt{\log \langle D \rangle} M_{\lambda(t_{j-1})}^{\delta, \kappa} f(t_{j-1})) dv \quad (t_j - t_{j-1}) \\
&+ N^2 \left( \sup_{\tau', \tau'' \in [0, T_1]} \|\log \langle D \rangle M_{\lambda(\tau')}^{\delta, \kappa} f(\tau'')\|_{L^2} \right)^2 O(|t_j - t_{j-1}|^2),
\end{aligned}$$

it follows from a similar formula as (4.8) that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \sum_{j=1}^k \int f(t_j) \left( (M_{\lambda(t_j)}^{\delta, \kappa})^2 - (M_{\lambda(t_{j-1})}^{\delta, \kappa})^2 \right) f(t_{j-1}) dv \\
&= N \int_0^t \int_{\mathbb{R}^3} (\sqrt{\log \langle D \rangle} M_{\lambda(\tau)}^{\delta, \kappa} f(\tau))^2 dv d\tau.
\end{aligned}$$

Summing up (4.9) with respect to  $j = 1, \dots, k$  and making  $k \rightarrow \infty$ , we obtain

$$\begin{aligned}
(4.10) & \frac{1}{2} \int_{\mathbb{R}^3} (M_{\lambda(t)}^{\delta, \kappa} f(t))^2 dv - \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} f(\tau) \left( \partial_\tau (M_{\lambda(\tau)}^{\delta, \kappa})^2 \right) f(\tau) dv d\tau \\
&= \frac{1}{2} \int_{\mathbb{R}^3} (M_{\lambda(0)}^{\delta, \kappa} f_0)^2 dv \\
&+ \int_0^t \left( Q(f(\tau), M_{\lambda(\tau)}^{\delta} f(\tau)), \frac{M_{\lambda(\tau)}^{\delta}}{(1 + \kappa \langle D \rangle)^2} f(\tau) \right)_{L^2} d\tau \\
&+ \int_0^t \left( M_{\lambda(\tau)}^{\delta} Q(f(\tau), f(\tau)) - Q(f(\tau), M_{\lambda(\tau)}^{\delta} f(\tau)), \frac{M_{\lambda(\tau)}^{\delta}}{(1 + \kappa \langle D \rangle)^2} f(\tau) \right)_{L^2} d\tau,
\end{aligned}$$

thanks to Proposition 3.6 and Theorem 3.5. In fact, for example, we have

$$\begin{aligned}
& \left| \left( Q(f(\tau), M_{\lambda(t_j)}^{\delta} f(\tau)), \frac{M_{\lambda(t_j)}^{\delta}}{(1 + \kappa \langle D \rangle)^2} f(t_j) \right)_{L^2} \right| \\
&\lesssim \sup_{\tau, t_j \in [0, T_1]} \left\{ \|f(\tau)\|_{L_{\gamma+2s}^1} \|M_{\lambda(t_j)}^{\delta} f(\tau)\|_{H_{\gamma/2+s}^s} \left\| \frac{M_{\lambda(t_j)}^{\delta}}{(1 + \kappa \langle D \rangle)^2} f(t_j) \right\|_{H_{\gamma/2+s}^s} \right\},
\end{aligned}$$

and hence the Lebesgue convergence theorem yields (4.10) because,

$$\left\| \frac{M_{\lambda(t_j)}^{\delta}}{(1 + \kappa \langle D \rangle)^2} f(t_j) \right\|_{H_{\gamma/2+s}^s} \rightarrow \left\| \frac{M_{\lambda(\tau)}^{\delta}}{(1 + \kappa \langle D \rangle)^2} f(\tau) \right\|_{H_{\gamma/2+s}^s} \quad \text{as } |t_j - \tau| \rightarrow 0.$$

Taking  $\kappa \rightarrow 0$  in (4.10) we obtain the desired formula. The last assertion of the lemma follows easily from the one of Theorem 3.5.  $\square$

## 5. SMOOTHING EFFECT OF $L^1$ WEAK SOLUTIONS

We come back to the proof of Theorem 1.2 starting from the  $L^1$  weak solution. The first part of the theorem is restated as follows:

**Theorem 5.1.** *Assume that  $0 < s < 1$ ,  $\gamma > \max\{-2s, -1\}$ . If  $f$  belongs to  $L^\infty([t_0, T]; L_\ell^1(\mathbb{R}^3))$  for any  $\ell \in \mathbb{N}$  and is a weak solution of (1.1), then for any  $t_0 < \tilde{t}_0 < T$ , we have*

$$f \in L^\infty([\tilde{t}_0, T]; \mathcal{S}(\mathbb{R}^3)).$$

*Proof.* By Theorem 4.1, it is sufficient to prove, for any  $0 < t_1 \leq T$ , (take again  $t_0 = 0$ )

$$(5.1) \quad f \in L^\infty([t_1, T]; L_\ell^2(\mathbb{R}^3)).$$

Since  $L^1(\mathbb{R}^3) \subset H^{-3/2-\varepsilon}$ , we assume that for any  $\ell$  and any  $0 < \varepsilon \ll 1$

$$(5.2) \quad \sup_{[0, T]} \|f(t, \cdot)\|_{H_\ell^{-3/2-\varepsilon}} < \infty.$$

As in the proof of Theorem 4.1, we shall prove the theorem by induction. Assume that for  $0 > a \geq -3/2 - \varepsilon$ , we have

$$\sup_{[0, T]} \|f(t, \cdot)\|_{H_\ell^a} < \infty.$$

Take also  $\lambda(t) = Nt + a$  for  $N > 0$ .

We first consider the case  $0 < s \leq 1/2$ . Choose  $N_0 = a + (5 + \gamma)/2 \geq 1 - \varepsilon + (\gamma/2) > 0$  such that (3.8) is fulfilled. Put  $\varepsilon_0 = (1 - 2s')/8 > 0$  and consider  $\varepsilon = \varepsilon_0$ , where  $0 < s' < s$  is chosen to satisfy  $\gamma + 2s' > 0$ . If we choose  $N, T_1 > 0$  such that  $NT_1 = \varepsilon_0$  then

$$s + \lambda(T_1) - N_0 - a = s + \varepsilon_0 - N_0 \leq s - 1 + 2\varepsilon_0 - (\gamma/2) < (s' - 1/2) + 2\varepsilon_0 < 0,$$

which shows

$$(5.3) \quad M_{\lambda(t)}^\delta f(t) \in L^\infty([0, T_1]; H_\ell^s(\mathbb{R}^3)).$$

This and Lemma 4.3 lead to (4.4), and hence we obtain (4.5) using Theorem 3.5, and (4.6) by means of (2.2) and Lemma 3.7. The same procedure as in the proof of Theorem 4.1 shows (5.1) by induction.

When  $s > 1/2$  we choose  $1/2 < s' < s$  such that  $\gamma + 2s' > 0$ ,  $2s' \geq (2s - 1)$ . Choose  $N_0 = a + (5 + \gamma + 2s' - 1)/2$  such that (3.9) is satisfied. Put  $\varepsilon_0 = (\gamma + 1)/10 > 0$  and consider  $\varepsilon = \varepsilon_0$ . Then, we have

$$(5.4) \quad s + \lambda(T_1) - N_0 - a = s + \varepsilon_0 - N_0 \leq s - s' + 2\varepsilon_0 - (1 + \gamma)/2 = s - s' - 3\varepsilon_0.$$

Since we may assume  $s - s' \leq \varepsilon_0$ , (5.4) also shows (5.3), which completes the proof of the theorem by the same way as in the case  $0 < s \leq 1/2$ .  $\square$

In view of Remark 4.2 and the last assertion of Lemma 4.3, the proof of Theorem 5.1 in the case  $0 < s < 1/2$  leads us easily to the following theorem where the assumption (1.5) can be removed.

**Theorem 5.2.** *Suppose that the cross section  $B$  of the form (1.2) satisfies (1.3) and (1.4) with  $0 < s < 1/2$  and  $\gamma > 0$ . If*

$$f \in L^\infty([0, T]; L_{\max\{2, \gamma/2+s\}}^1(\mathbb{R}^3) \cap L \log L) \cap L^1([0, T]; L_{2+\gamma}^1(\mathbb{R}^3))$$

*is a weak solution, then  $f \in L^\infty([t_0, T]; H^\infty(\mathbb{R}^3))$  for any  $t_0 \in ]0, T[$ .*

We consider now the second part of Theorem 1.2, which is stated as follows:

**Theorem 5.3.** *Assume that  $-1 \geq \gamma > -2s$ . Let  $f \in L^\infty([t_0, T]; L_\ell^1(\mathbb{R}^3))$  for any  $\ell \in \mathbb{N}$  be a weak solution of (1.1) satisfying the entropy dissipation estimate*

$$\int_{t_0}^T D(f(t), f(t)) dt < +\infty.$$

*Then for any  $t_0 < \tilde{t}_0 < T$ , we have*

$$f \in L^\infty([\tilde{t}_0, T]; \mathcal{S}(\mathbb{R}^3)).$$

For the proof, we only need to reconsider the term  $A_1$  defined in (3.12) under the hypothesis  $-1 \geq \gamma > -2s$ . Note that we can now choose arbitrarily large  $N_0$  in (3.1) because neither (3.8) nor (3.9) is required. Hence  $(M_{\lambda(t)}^\delta)^2 f(t)$  belongs to  $W^{2,\infty}$ , which enable us to take  $(M_{\lambda(t)}^\delta)^2 f(t)$  as a test function. However  $\lambda(t)$  can not be taken as large as we want, because it is also restricted to the small gain regularity coming from the dissipation estimate. Thanks to Theorem 4.1, it suffices to show  $f \in L^\infty([T_0, T_1]; L_\ell^2)$  by induction, starting from (5.2) where we take again  $t_0 = 0$ .

It follows from (3.3) of Lemma 3.1, (3.10) and (3.11) that  $A_1$  can be replaced by

$$(5.5) \quad \tilde{A}_{1,\lambda} = \iint_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma}} |\hat{g}(\xi - \xi_*)| |\hat{h}(\xi)| \langle \xi \rangle^\lambda \mathbf{1}_{\langle \xi_* \rangle \geq \sqrt{2}|\xi|} \frac{\langle \xi \rangle}{\langle \xi_* \rangle} d\xi d\xi_*$$

We divide the proof in three steps.

**1st step:** Take  $s' > 1/2$  such that  $\gamma + 2s' > 0$  and  $s' < s$ . Put  $s_0 = \frac{1}{4}(\gamma + 2s')$ . For arbitrary  $t > 0$  and  $N > 0$  satisfying  $Nt = s_0$ , we set

$$\lambda_1(\tau) = N\tau - \frac{3}{2} - \varepsilon \text{ for } \tau \in [0, t],$$

where  $\varepsilon > 0$  is arbitrarily small. If we substitute  $\lambda = \lambda_1(\tau)$  into (5.5) then, in view of  $N\tau \leq s_0$ , we have

$$\begin{aligned} \tilde{A}_{1,\lambda_1}(\tau) &\lesssim \|\hat{g}\|_{L^\infty} \iint_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma-s_0}} \frac{|\hat{h}(\xi)|}{\langle \xi \rangle^{3/2+\varepsilon}} d\xi d\xi_* \\ &\lesssim \|\hat{f}\|_{L^{3/(2s')}} \|g\|_{L^1} \|h\|_{L^2} \lesssim \|f\|_{L^{3/(3-2s')}} \|g\|_{L^1} \|h\|_{L^2} \end{aligned}$$

because of the Hölder inequality and the fact that  $(3 + \gamma - s_0)\{3/(3 - 2s')\} > 3$ . By means of Lemma 3.8, we have for some  $\ell_0 > 0$

$$\begin{aligned} \tilde{A}_{1,\lambda_1} &\lesssim \left( \|\langle v \rangle^\gamma f\|_{L^{3/(3-2s)}} + \|f\|_{L_{\ell_0}^1} \right) \|g\|_{L^1} \|h\|_{L^2} \\ &\lesssim \left( \|\langle v \rangle^{\gamma/2} \sqrt{f}\|_{H^s}^2 + \|f\|_{L_{\ell_0}^1} \right) \|g\|_{L^1} \|h\|_{L^2}. \end{aligned}$$

Putting  $f = g = f(\tau, v)$  and  $h = M_{\lambda_1(\tau)}^\delta f(\tau, v)$ , we have a term coming from  $\tilde{A}_{1,\lambda_1}$  in estimating

$$\int_0^t \left( M_{\lambda_1}^\delta Q(f(\tau), f(\tau)) - Q(f(\tau), M_{\lambda_1}^\delta f(\tau)), M_{\lambda_1}^\delta f(\tau) \right) d\tau$$

as follows:

$$\begin{aligned}
& \left( \sup_{\tau \in [0, t]} \|f(\tau)\|_{L^1} \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{L^2} \right) \int_0^t \|\langle v \rangle^{\gamma/2} \sqrt{f(\tau)}\|_{H^s}^2 d\tau \\
& + \left( \sup_{\tau \in [0, t]} \|f(\tau)\|_{L_{\ell_0}^1}^2 \right) \sqrt{t} \left( \int_0^t \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{L^2}^2 d\tau \right)^{1/2} \\
& \leq \frac{1}{10} \sup_{\tau \in [0, t]} \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{L^2}^2 + C_f \left\{ \left( \int_0^t D(f(\tau), f(\tau)) d\tau \right)^2 \right. \\
& \quad \left. + t \sup_{\tau \in [0, t]} \|f(\tau)\|_{L_{\ell_0}^1}^4 + \int_0^t \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{L^2}^2 d\tau \right\},
\end{aligned}$$

where we have used Corollary 2.4 in the last inequality. Instead of (4.6), we obtain

$$\begin{aligned}
& \|M_{\lambda_1(t)}^\delta f(t)\|_{L^2}^2 - \frac{1}{10} \sup_{\tau \in [0, t]} \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{L^2}^2 + \int_0^t \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{H^{s'}}^2 d\tau \\
& \lesssim \|M_{\lambda_1(0)}^\delta f(0)\|_{L^2}^2 + \left( \int_0^t D(f(\tau), f(\tau)) d\tau \right)^2 \\
& \quad + t \sup_{\tau \in [0, t]} \|f(\tau)\|_{L_{\ell_0}^1}^4 + \int_0^t \|f(\tau)\|_{H_\ell^2}^2 d\tau.
\end{aligned}$$

If we consider  $\tau \in [0, t]$  instead of  $t$  then the first term on the right hand side can be replaced by  $\sup_{\tau \in [0, t]} \|M_{\lambda_1(\tau)}^\delta f(\tau)\|_{L^2}^2$ , which absorbs the second term on the right hand side. Letting  $\delta \rightarrow 0$  we obtain, in view of  $Nt = s_0$ ,

$$(5.6) \quad \|\langle D \rangle^{s_0-3/2-\varepsilon} f(t)\|_{L^2} < \infty,$$

and

$$(5.7) \quad \int_0^t \|\langle D \rangle^{N\tau-3/2-\varepsilon} f(\tau)\|_{H^{s'}}^2 d\tau < \infty.$$

**2nd step:** Let  $\kappa > 0$  be small arbitrarily. Considering  $\tau \in [\kappa, t]$  instead of  $t$  in (5.6), we may assume

$$\sup_{\tau \in [\kappa, t]} \|\langle D \rangle^{s_0-3/2-\varepsilon} f(\tau)\|_{L^2} < \infty.$$

For arbitrary  $t > \kappa$  and  $N > 0$  satisfying  $N(t - \kappa) = s_0$  we set

$$\lambda_2(\tau) = s_0 + N(\tau - \kappa) - \frac{3}{2} - \varepsilon \text{ for } \tau \in [\kappa, t].$$

If we substitute  $\lambda = \lambda_2(\tau)$  into (5.5) then we have

$$\begin{aligned}
\tilde{A}_{1, \lambda_2}(\tau) & \lesssim \int_{\mathbb{R}^3} \frac{|\langle \xi \rangle^{s'} \hat{h}(\xi)|}{\langle \xi \rangle^{3/2+\varepsilon}} \\
& \left( \int_{\mathbb{R}^3} \frac{\langle \xi_* \rangle^{s_0-\frac{3}{2}-\varepsilon} |\hat{f}(\xi_*)| \langle \xi_* - \xi \rangle^{N(\tau-\kappa)-\frac{3}{2}-\varepsilon+s'} |\hat{g}(\xi - \xi_*)|}{\langle \xi_* \rangle^{3+\gamma+2s'-3-2\varepsilon}} \left( \frac{\langle \xi \rangle}{\langle \xi_* \rangle} \right)^{1-s'} d\xi_* \right) d\xi \\
& \lesssim \|f\|_{H^{s_0-\frac{3}{2}-\varepsilon}} \|\langle D \rangle^{s'+N(\tau-\kappa)-\frac{3}{2}-\varepsilon} g\|_{L^2} \|h\|_{H^{s'}},
\end{aligned}$$

if  $\gamma + 2s' > 2\varepsilon$ . Putting  $f = g = f(\tau, v)$  and  $h = M_{\lambda_2(\tau)}^\delta f(\tau, v)$  we have a term coming from  $\tilde{A}_{1,\lambda_2}$  in estimating

$$\left| \int_{\kappa}^t \left( M_{\lambda_2}^\delta Q(f(\tau), f(\tau)) - Q(f(\tau), M_{\lambda_2}^\delta f(\tau)), M_{\lambda_2}^\delta f(\tau) \right) d\tau \right|$$

as follows:

$$\begin{aligned} & \left( \sup_{\tau \in [\kappa, t]} \|f(\tau)\|_{H^{s_0 - \frac{3}{2} - \varepsilon}} \right) \left\{ \int_{\kappa}^t \|M_{\lambda_2(\tau)}^\delta f(\tau)\|_{H^{s'}}^2 d\tau \right\} \\ & + \int_{\kappa}^t \|\langle D \rangle^{s' + N(\tau - \kappa) - \frac{3}{2} - \varepsilon} f(\tau)\|_{L^2}^2 d\tau \}. \end{aligned}$$

In order to avoid the confusion we write  $N = N_2 = s_0/(t - \kappa)$  in this second step and  $N = N_1 = s_0/t$  in (5.7). Then we have

$$N_2(\tau - \kappa) \leq N_1\tau \quad \text{if } \tau \in [\kappa, t],$$

from which we can use (5.7) to estimate the term coming from  $\tilde{A}_{1,\lambda_2}$ . In this step we finally obtain, in view of  $N(t - \kappa) = s_0$ ,

$$\|\langle D \rangle^{2s_0 - 3/2 - \varepsilon} f(t)\|_{L^2} < \infty$$

and

$$(5.8) \quad \int_{\kappa}^t \|\langle D \rangle^{s_0 - 3/2 - \varepsilon} f(\tau)\|_{H^{s'}}^2 d\tau < \infty.$$

**3rd step:** For  $k \geq 2$ , suppose that

$$\sup_{\tau \in [(k-1)\kappa, t]} \|\langle D \rangle^{(k-1)s_0 - 3/2 - \varepsilon} f(\tau)\|_{L^2} < \infty.$$

For arbitrary  $t > k\kappa$  and  $N > 0$  satisfying  $N(t - k\kappa) = s_0$  we set

$$\lambda_k(\tau) = (k-1)s_0 + N(\tau - \kappa) - \frac{3}{2} - \varepsilon \quad \text{for } \tau \in [\kappa, t].$$

Consider  $M_{\lambda_k(\tau)}^\delta$ . Then, using (5.8) instead of (5.7), we can proceed the induction method by the almost same way as in the second step. Since  $\kappa > 0$  is arbitrary we obtain the desired conclusion.

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